Linear Algebra I
Problem Sheet 3: Inner Products

Inner Product Spaces

Question 1. Let V be the vector space of polynomials of degree less than 4 with real coefficients. This is a vector space over $\mathbb{R}$.

1. Prove that $(V, \langle , \rangle)$ is an inner product space with inner product defined for all f, $g \in V$ by
$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$
2. Produce an orthonormal basis for $(V, \langle , \rangle)$ by applying the Gram-Schmidt orthonormalisation procedure to the basis $\{1, x, x^2, x^3\}$ of V.

Question 2. Decide in any inner product space whether n vectors $v_1, \ldots, v_n$ being mutually orthogonal ($\langle v_i, v_j \rangle = 0$ for all $i \neq j$) means $\{v_1, \ldots, v_n\}$ is linearly independent.

Question 3. Why does the notion of an inner product break down for a finite field? To illustrate your answer, for a vector space V over the finite field $\mathbb{Z}_2$, try to construct an operation
$$f : V \times V \to \mathbb{Z}_2$$
satisfying the four properties of an inner product.

Question 4. Consider the linear map
$$L : \mathbb{R}^4 \to \mathbb{R}^4 : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mapsto \begin{pmatrix} x + w \\ x + y + 2w \\ y + z \\ x + z \end{pmatrix}.$$
Find the orthogonal complement $\mathrm{Im}(L)^\perp$ of $\mathrm{Im}(L)$. Use the Gram-Schmidt orthonormalisation procedure to find an orthonormal basis for $\mathrm{Im}(L)$.